Utility Option Pricing Model (UOPM)
Two-State Model - Option Pricing in Incomplete Markets

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In this white paper we will price a put and call option assuming that markets are incomplete. Since the criteria for a complete market are violated, all assets are not tradeable and arbitrages are possible such that pricing options via no-arbitrage assumptions is no longer viable. We will work through the following hypothetical problem from Part I...

Our Hypothetical Problem

We have two possible states of the world (SOTW) at the end of the option term: SOTW = Up and SOTW = Down. We are given the following model assumptions... [1]

Table 1: Model Assumptions From Part I

Symbol	Definition	Value
S_0	Share price at time zero ($)	20.00
B_0	Risk-free bond price at time zero ($)	100.00
X_T	Option exercise price at time T ($)	22.50
α	Bond continuous-time risk-free rate (%)	4.15
μ	Share continous-time return drift (%)	7.50
σ	Share continous-time return volatility (%)	18.00
p	Probability that share price will increase (%)	50.00
$(1-p)$	Probability that share price will decrease (%)	50.00
T	Option term in years (#)	3.00

Table 2: End-Of-Term Random Asset Prices From Part I

Symbol	Description	Value
$S(U)$	Share price given SOTW = Up	34.21
$S(D)$	Share price given SOTW = Down	18.34
$B(U)$	Bond price given SOTW = Up	113.26
$B(D)$	Bond price given SOTW = Down	113.26
$C(U)$	Call option price given SOTW = Up	11.71
$C(D)$	Call option price given SOTW = Down	0.00
$P(U)$	Put option price given SOTW = Up	0.00
$P(D)$	Put option price given SOTW = Down	4.16

Questions:
Question 1: What is expected call option price and put option price at time T.
Question 2: Calculate the coefficient of risk aversion.
Question 3: What are the expected utilities of option payoffs at time T?
Question 4: What are option prices at time zero?
Question 5: How would the Black-Scholes OPM price these options?

Option Price Equations

We will define the variable X_T to be the option exercise price at time T. This statement in equation form is...

$$X_T = \text{Option exercise price} \tag{1}$$

We will define the variables $C(U)$ and $C(D)$ to be call option payoffs at time T given that share price increases or decreases, respectively, over the time interval $[0, T]$. Using the data in Tables 1 and 2 above, the equations for call option payoffs are...

$$C(U) = \text{Max}\left[S(U) - X_T, 0\right] \text{ ...and... } C(D) = \text{Max}\left[S(D) - X_T, 0\right] \tag{2}$$

We will define the variables $P(U)$ and $P(D)$ to be put option payoffs at time T given that share price increases or decreases, respectively, over the time interval $[0, T]$. Using the data in Tables 1 and 2 above, the equations for put option payoffs are...

$$P(U) = \text{Max}\left[X_T - S(U), 0\right] \text{ ...and... } P(D) = \text{Max}\left[X_T - S(D), 0\right] \tag{3}$$

We will define the variables C_T and P_T to be the call option and put option price, respectively, at time T. Using Equations (2) and (3) above and the data in Tables 1 and 2 above, the equations for expected option prices at time T is...

$$\mathbb{E}\left[C_T\right] = p\,C(U) + (1-p)\,C(D) \text{ ...and... } \mathbb{E}\left[P_T\right] = p\,P(U) + (1-p)\,P(D) \tag{4}$$

Option Pricing in Incomplete Markets

We will define the variable λ to be average investor's coefficient of risk aversion. The equation for the exponential untility function given the i'th state of the world is...

$$U(W_i) = 1 - \text{Exp}\left\{-\lambda\,W_i\right\} \text{ ...where... } W_i = \text{Wealth at time } T \Big/ \text{Share price at time zero} \tag{5}$$

There are two states of the world at time T, share price increases ($S(U)$) and share price decreases ($S(D)$). Using Equation (5) above, the equation for expected utility of scaled wealth at time T is...

$$\mathbb{E}\left[U(W)\right] = p\left(1 - \text{Exp}\left\{-\lambda\,W(U)\right\}\right) + (1-p)\left(1 - \text{Exp}\left\{-\lambda\,W(D)\right\}\right) \tag{6}$$

Using Equation (**??**) above, the equations for scaled wealth at time T are...

$$W(U) = \frac{S(U)}{S_0} = \text{Exp}\left\{\mu\,T + \sigma\sqrt{T}\right\} \text{ ...and... } W(D) = \frac{S(D)}{S_0} = \text{Exp}\left\{\mu\,T - \sigma\sqrt{T}\right\} \tag{7}$$

Using Equation (7) above, we can rewrite Equation (6) above as...

$$\mathbb{E}\left[U(W)\right] = p\left(1 - \text{Exp}\left\{-\lambda\,\text{Exp}\left\{\mu\,T + \sigma\sqrt{T}\right\}\right\}\right) + (1-p)\left(1 - \text{Exp}\left\{-\lambda\,\text{Exp}\left\{\mu\,T - \sigma\sqrt{T}\right\}\right\}\right) \tag{8}$$

Note that we can rewrite Equation (8) above as...

$$\mathbb{E}\left[U(W)\right] = 1 - p\,\text{Exp}\left\{-\lambda\,\text{Exp}\left\{\mu\,T + \sigma\sqrt{T}\right\}\right\} - (1-p)\,\text{Exp}\left\{-\lambda\,\text{Exp}\left\{\mu\,T - \sigma\sqrt{T}\right\}\right\} \tag{9}$$

Solving For The Risk Aversion Coefficient

We will define the variable $CE(S)$ to be the certainty equivalent of random share prices at time T. The dollar value of the certainty equivalent is such that the utility of the scaled certainty equivalent is equal to the utility of expected scaled wealth. This statement in equation form is... [2]

$$U\left(\frac{CE(S)}{S_0}\right) = \mathbb{E}\left[U(W)\right] \tag{10}$$

The equation for the certainty equivalent of share price at time T is... [2]

$$CE(S)=S_0\,\text{Exp}\left\{\alpha\,T\right\}\ \text{...where...}\ \frac{CE(S)}{S_0}=\text{Exp}\left\{\alpha\,T\right\}\ \text{...and...}\ U\left(\frac{CE(S)}{S_0}\right)=1-\text{Exp}\left\{-\lambda\,\text{Exp}\left\{\alpha\,T\right\}\right\}\tag{11}$$

Using Equations (9) and (11) above, we can rewrite Equation (10) above as...

$$1-p\,\text{Exp}\left\{-\lambda\,\text{Exp}\left\{\mu\,T+\sigma\sqrt{T}\right\}\right\}-(1-p)\,\text{Exp}\left\{-\lambda\,\text{Exp}\left\{\mu\,T-\sigma\sqrt{T}\right\}\right\}-\left(1-\text{Exp}\left\{-\lambda\,\text{Exp}\left\{\alpha\,T\right\}\right\}\right)=0\tag{12}$$

Using Equation (12) above, we will define the function $f(\lambda)$ to be...

$$f(\lambda)=0\ \text{...where...}\ \lambda=\text{Actual coefficient of risk-aversion}\tag{13}$$

Using Appendix Equations (27) and (28) below, we will define the function $f(\hat{\lambda})$ to be...

$$f(\hat{\lambda})=\text{Exp}\left\{-\lambda\,C\right\}-p\,\text{Exp}\left\{-\lambda\,A\right\}-(1-p)\,\text{Exp}\left\{-\lambda\,B\right\}\ \text{...where...}$$
$$A=\text{Exp}\left\{\mu\,T+\sigma\sqrt{T}\right\}\ \text{...and...}\ B=\text{Exp}\left\{\mu\,T-\sigma\sqrt{T}\right\}\ \text{...and...}\ C=\text{Exp}\left\{\alpha\,T\right\}\tag{14}$$

Using Appendix Equations (27) and (29) below, the derivative of Equation (14) above with respect to the λ is...

$$f'(\hat{\lambda})=-C\,\text{Exp}\left\{-\lambda\,C\right\}+p\,A\,\text{Exp}\left\{-\lambda\,A\right\}+(1-p)\,B\,\text{Exp}\left\{-\lambda\,B\right\}$$
$$A=\text{Exp}\left\{\mu\,T+\sigma\sqrt{T}\right\}\ \text{...and...}\ B=\text{Exp}\left\{\mu\,T-\sigma\sqrt{T}\right\}\ \text{...and...}\ C=\text{Exp}\left\{\alpha\,T\right\}\tag{15}$$

To solve for λ, the Newton-Raphson equation that we will iterate is... [3]

$$\lambda+\hat{\epsilon}=\hat{\lambda}+\frac{f(\lambda)-f(\hat{\lambda})}{f'(\hat{\lambda})}\ \Bigg|\ f(\lambda)=\text{Equation (13)},\ f(\hat{\lambda})=\text{Equation (14)},\ f'(\hat{\lambda})=\text{Equation (15)}\tag{16}$$

Calculating Option Price

We will define the variable $O(U)$ and $O(D)$ to option payoffs at time T given that share price increases or decreases, respectively. Given that we are long the option, the probability-weighted option payoffs at time T are...

$$\text{Receive } O(U) \text{ with probability } p\ \text{...or...}\ \text{Receive } O(D) \text{ with probability } 1-p\tag{17}$$

We will define the function $U(O)$ to be the utility of the option payoff at time T given the state of the world at that time. Using Equations (5) and (17) above, the equation for the expected utility of scaled option payoffs at time T is...

$$\mathbb{E}\Big[U(O)\Big]=p\left(1-\text{Exp}\left\{-\lambda\,\frac{O(U)}{S_0}\right\}\right)+(1-p)\left(1-\text{Exp}\left\{-\lambda\,\frac{O(D)}{S_0}\right\}\right)\tag{18}$$

We will define the $CE(O)$ to be the certainty equivalent of option payoffs at time T and the variable O_0 to be option price at time zero. Using Equation (11) above as our guide, the equation for the certainty equivalent is...

$$\text{if...}\ CE(O)=O_0\,\text{Exp}\left\{\alpha\,T\right\}\ \text{...then...}\ U\left(\frac{CE(O)}{S_0}\right)=1-\text{Exp}\left\{-\lambda\,\frac{O_0}{S_0}\,\text{Exp}\left\{\alpha\,T\right\}\right\}\tag{19}$$

Note that the definition of certainty equivalents requires that the following equations holds...

$$U\left(\frac{CE(O)}{S_0}\right)=\mathbb{E}\Big[U(O)\Big]\tag{20}$$

Using Equations (18) and (19) above, we can rewrite Equation (20) above as...

$$1-\text{Exp}\left\{-\lambda\,\frac{O_0}{S_0}\,\text{Exp}\left\{\alpha\,T\right\}\right\}=\mathbb{E}\Big[U(C)\Big]\tag{21}$$

Using Appendix Equation (30) below, the solution to Equation (21) above is...

$$O_0=-S_0\,\ln\left(1-\mathbb{E}\Big[U(O)\Big]\right)\text{Exp}\left\{-\alpha\,T\right\}\lambda^{-1}\tag{22}$$

Answers To Our Hypothetical Problem

Question 1: What is expected call option price and put option price at time T.

Using Equation (4) above and the data in Tables 1 and 2 above, the answer to the question is...

$$\mathbb{E}[C_T] = 0.50 \times 11.71 + (1 - 0.50) \times 0.00 = 5.85 \text{ ...and... } \mathbb{E}[P_T] = 0.50 \times 0.00 + (1 - 0.50) \times 4.16 = 2.08$$

Question 2: Calculate the coefficient of risk aversion.

After iterating Equation (16) above, the coefficient of risk aversion for our two state asset price model is...

Iteration	$\hat{\lambda}$	$f(\lambda)$	$f(\hat{\lambda})$	$f'(\hat{\lambda})$
0	3.000000	0.000000	-0.001448	-0.003543
1	2.591204	0.000000	0.000732	-0.007416
2	2.689870	0.000000	0.000055	-0.006317
3	2.698604	0.000000	0.000000	-0.006225
4	2.698669	0.000000	0.000000	-0.006224
5	2.698669	0.000000	0.000000	-0.006224

The answer to the question is $\lambda = 2.698669$ (2.70 rounded).

Note that the coefficient of risk aversion is commonly thought to be between 2.00 and 4.00 so therefore our initial guess value was the average of these two numbers.

Question 3: What are the expected utilities of option payoffs at time T?

Using Equation (18) above and the data in Tables 1 and 2 above, the expected utility of call option payoffs at time T is...

$$\mathbb{E}\left[U(C)\right] = 0.50 \times \left(1 - \text{Exp}\left\{ - 2.70 \times \frac{11.71}{20.00} \right\}\right) + (1 - 0.50) \times \left(1 - \text{Exp}\left\{ - 2.70 \times \frac{0.00}{20.00} \right\}\right) = 0.3970 \qquad (23)$$

Using Equation (18) above and the data in Tables 1 and 2 above, the expected utility of put option payoffs at time T is...

$$\mathbb{E}\left[U(P)\right] = 0.50 \times \left(1 - \text{Exp}\left\{ - 2.70 \times \frac{0.00}{20.00} \right\}\right) + (1 - 0.50) \times \left(1 - \text{Exp}\left\{ - 2.70 \times \frac{4.16}{20.00} \right\}\right) = 0.2149 \qquad (24)$$

Question 4: What are option prices at time zero?

Using Equation (22) above, call option price at time zero is...

$$C_0 = -20.00 \times \ln\left(1 - 0.3970\right) \times \text{Exp}\left\{ - 0.0415 \times 3.00 \right\} \times 2.70^{-1} = 3.31 \qquad (25)$$

Using Equation (22) above, put option price at time zero is...

$$P_0 = -20.00 \times \ln\left(1 - 0.2149\right) \times \text{Exp}\left\{ - 0.0415 \times 3.00 \right\} \times 2.70^{-1} = 1.58 \qquad (26)$$

Question 5: How would the Black-Scholes OPM price these options?

The option values via our two-state model (UOPM) and the Black-Scholes model (BSOPM) are...

Description	UOPM	BSOPM
Call option	3.31	2.54
Put option	1.58	2.40

Appendix

A. We will make the following definitons...

$$A = \mathrm{Exp}\left\{\mu\,T + \sigma\sqrt{T}\right\} \text{ ...and... } B = \mathrm{Exp}\left\{\mu\,T - \sigma\sqrt{T}\right\} \text{ ...and... } C = \mathrm{Exp}\left\{\alpha\,T\right\} \tag{27}$$

B. Using the definitions in Equation (27) above, we can rewrite the left side of Equation (12) above as...

$$\mathrm{Exp}\left\{-\lambda\,C\right\} - p\,\mathrm{Exp}\left\{-\lambda\,A\right\} - (1-p)\,\mathrm{Exp}\left\{-\lambda\,B\right\} \tag{28}$$

C. The equation for the derivative of Equation (28) above with respect to λ is...

$$-C\,\mathrm{Exp}\left\{-\lambda\,C\right\} + p\,A\,\mathrm{Exp}\left\{-\lambda\,A\right\} + (1-p)\,B\,\mathrm{Exp}\left\{-\lambda\,B\right\} \tag{29}$$

D. The solution to the following equation in terms of option price at time zero is...

$$1 - \mathrm{Exp}\left\{-\lambda\,\frac{O_0}{S_0}\,\mathrm{Exp}\left\{\alpha\,T\right\}\right\} = \mathbb{E}\left[U(O)\right]$$

$$1 - \mathbb{E}\left[U(O)\right] = \mathrm{Exp}\left\{-\lambda\,\frac{O_0}{S_0}\,\mathrm{Exp}\left\{\alpha\,T\right\}\right\}$$

$$\ln\left(1 - \mathbb{E}\left[U(O)\right]\right) = -\lambda\,\frac{O_0}{S_0}\,\mathrm{Exp}\left\{\alpha\,T\right\}$$

$$O_0 = -S_0\,\ln\left(1 - \mathbb{E}\left[U(O)\right]\right)\mathrm{Exp}\left\{-\alpha\,T\right\}\lambda^{-1} \tag{30}$$

References

[1] Gary Schurman, *Utility Option Pricing Model (UOPM) - The Two-State Asset Model,* December, 2023.

[2] Gary Schurman, *Utility Functions: Case Study - Investor Risk Aversion Coefficient,* November, 2023.

[3] Gary Schurman, *The Newton-Raphson Method For Solving Non-Linear Equations,* October, 2009.